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Strong Chromatic Index of Graphs

summary of PhD dissertation

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1 Introduction

A **strong edge-coloring** of a graph G is a coloring of edges of G such that each color class is an induced matching (that is, no edge of G can intersect two edges of the same color). The **strong chromatic index** of G , denoted $s'(G)$, is the minimum number of colors in a strong edge-coloring of G .

Strong edge-coloring is a model of channel assignment in a wireless network, where the reachability relation is symmetric, i.e. if node u is in range of node v , then node v is also in range of node u (see [2] and [12]). We assume that each pair of nodes that are in range of each other will communicate and the goal is to assign channels so that all the communication can take place concurrently.

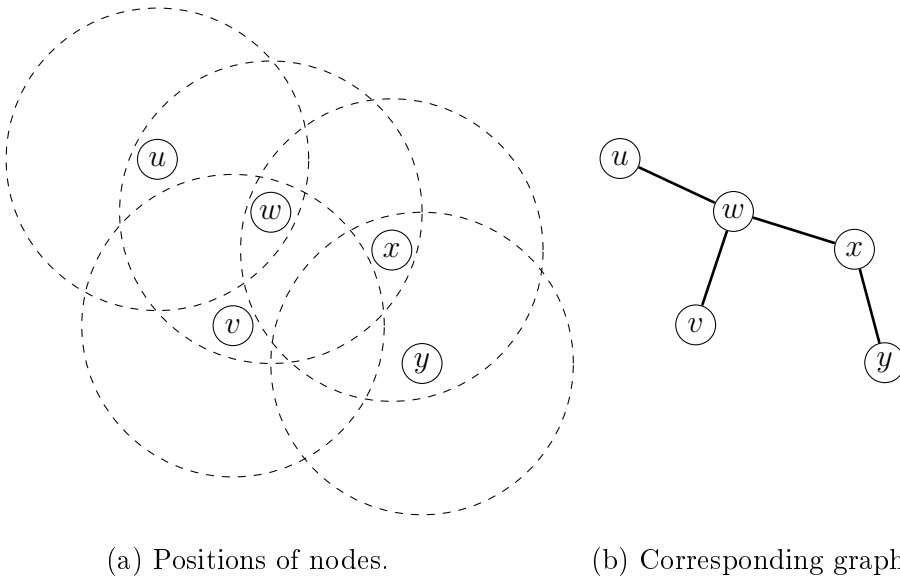


Figure 1: Model of communication in a wireless network.

To guarantee a successful communication, the channel assignment must satisfy the following constraint: if nodes u and v communicate on some channel c , then all other nodes that are in range of u and v can not use channel c (as otherwise either u or v would receive two interfering transmissions). Such an assignment corresponds to a strong edge-coloring of some graph G (where colors correspond to channels, vertices of G correspond to nodes and edges of G join pairs of nodes that are in range of each other), see Figure 1.

The dissertation revolves around one fundamental question: given a number Δ , what is the maximum possible strong chromatic index of a graph with maximum degree Δ ? It is possible to construct a graph G with maximum degree Δ such that $s'(G) = \frac{5}{4}\Delta^2$; in 1985, Erdős and Nešetřil conjectured that it is the worst possible.

Conjecture 1 (Erdős and Nešetřil, 1985 [8]). *If G is a graph with maximum degree Δ , then*

$$s'(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{for even } \Delta, \\ \frac{5}{4}\Delta^2 - \frac{2\Delta-1}{4}, & \text{for odd } \Delta. \end{cases}$$

The conjecture of Erdős and Nešetřil is far from being confirmed. It is easy to show, using a greedy algorithm, that $s'(G) \leq 2\Delta^2$, but even a tiny improvement of the leading constant is nontrivial. In 1997, Molloy and Reed proved, using a nonconstructive argument, that $s'(G) \leq 1.998\Delta^2$ [10], and this result was improved only recently by Bruhn and Joos to $s'(G) \leq 1.93\Delta^2$ [3], which remains the best known.

Our considerations also concern restricted and relaxed variants of the problem. In particular, we investigate the fractional strong edge-colorings (that, in terms of our motivation, correspond to communication with time-division) and a topological counterpart of strong chromatic index. We restrict our attention to specific classes of graphs that include bipartite graphs, chordless graphs and locally sparse graphs.

2 Methodology

The problem is tackled combinatorially, using a mixture of constructive methods (that yield polynomial-time algorithms) and nonconstructive approach. A common trait of our proofs is that they rely on low local density of considered graphs – where the exact meaning of “density” varies depending on the context.

We use a number of tools that concern chromatic number and related invariants of graphs: for a graph G , the chromatic number of G is denoted $\chi(G)$, the fractional chromatic number of G – $\chi_f(G)$, and the topological (counterpart of) chromatic number of G – $\chi_t(G)$.

The first tool is a fractional relaxation of Reed's conjecture, confirmed to be true by Molloy and Reed in 2002. It allows to deduce a nontrivial upper bound on $\chi_f(G)$ from the bound on the size of a maximum clique in a graph G .

Theorem 2 (Molloy and Reed, 2002 [11, Theorem 21.7]). *Let G be a graph with maximum degree Δ and maximum clique of size ω . We have*

$$s'_f(G) \leq \frac{\Delta + 1 + \omega}{2}.$$

Our second tool is a theorem of Alon, Krivelevich and Sudakov, which says that if the neighborhoods in a graph G are sparse (that is, they span asymptotically less than $\Delta(G)^2$ edges), then $\chi(G)$ is asymptotically smaller than $\Delta(G)$.

Theorem 3 (Alon, Krivelevich and Sudakov, 1999 [1]). *There exists a constant c such that the following is true. Let G be a graph with maximum degree Δ such that for every vertex $v \in V(G)$ the subgraph of G induced by $N(v)$ has at most $\frac{\Delta^2}{f}$ edges, where $1 < f \leq \Delta$. Then the chromatic number of G is at most $c \frac{\Delta}{\ln f}$.*

Finally, we use a theorem of Csorba, Lange, Schurr and Wassmer that allows to bound $\chi_t(G)$ using a purely combinatorial property of the graph G .

Theorem 4 (Csorba, Lange, Schurr and Wassmer, 2004 [4]). *If G is a graph satisfying $\chi_t(G) \geq t$, then for every possible $l, m \in \mathbb{N}$ with $l + m = t$, the complete bipartite graph $K_{l,m}$ appears as a subgraph of G .*

3 Results

The main result concerns the fractional strong chromatic index (denoted $s'_f(G)$ for a graph G) of bipartite graphs. A straightforward application of Theorem 2 to this problem gives the bound $s'_f(G) \leq 1.5\Delta^2$ (for a bipartite graph G of maximum degree Δ) and our contribution is that the constant 1.5 is not optimal and can be improved by at least a bit.

Theorem 5 (MD, 2015+ [5]). *Let G be a bipartite graph of maximum degree Δ . We have*

$$s'_f(G) \leq \frac{31}{21}\Delta^2 + \Delta^{1.5}.$$

Our second contribution is a bound on the strong chromatic index of chordless graphs (a graph G is chordless if every cycle in G is induced). The proof is constructive and yields a polynomial-time algorithm that finds the desired coloring; it is a 4-approximation, since we have $s'(G) \geq \Delta$.

Theorem 6 (MD, Grytczuk, Śleszyńska-Nowak, 2015 [7]). *If G is a chordless graph of maximum degree Δ , then*

$$s'(G) \leq 4\Delta - 3.$$

The next result can be thought of as a strengthening of Mahdian's theorem (that if G is a C_4 -free graph of maximum degree Δ , then we have $s'(G) \leq 2\frac{\Delta^2}{\log \Delta}$, see [9]); using a different technique we are able to show (asymptotically) the same bound even if we allow a small number of 4-cycles in a graph. Moreover, the proof yields a randomized algorithm that constructs the coloring.

Theorem 7 (MD, 2015+). *There exists a constant K such that the following holds. Let G be a graph of maximum degree Δ such that every edge of G is in at most $\frac{\Delta^2}{g}$ cycles of length 4, where $1 < g \leq \Delta^2$. We have*

$$s'(G) \leq K \frac{\Delta^2}{\ln g}.$$

We also investigate the topological strong chromatic index (denoted $s'_t(G)$ for a graph G). We show that for a bipartite graph G of maximum degree Δ we have $s'_t(G) \leq 1.703\Delta^2$, which is a significant improvement over the previous best bound of $1.93\Delta^2$.

Theorem 8 (MD, 2015 [6]). *Let G be a bipartite graph of maximum degree Δ . We have*

$$s'_t(G) \leq 1.703\Delta^2.$$

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